Non-linear Gagliardo–Nirenberg inequality involving a second-order elliptic operator in non-divergent form

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General concept of the Gagliardo–Nirenberg inequality may be seen as an estimate of the term based on the intermediate derivative by the term depending on both higher and lower derivative or rather by the higher derivative and the function itself, in the form

$$A(D^k f) \lesssim B(D^m f, D^l f), \text{ for } 0 \leq l < k < m.$$

Most of the variants of G–N inequality is in the linear form with term A being the norm or seminorm and B is the product of powers of norms or seminorms, in the form

$$\|D^k f\|_X \lesssim \|D^m f\|_Y^y \|D^l f\|_Z^z, \text{ for } 0 \leq l < k < m.$$

We introduce the following nonlinear weighted G–N inequality in the form

$$\int_\Omega |\nabla u(x)|^2 h(u(x))\,\mathrm{d}x \leq C \int_\Omega \left(\sqrt{|Pu(x)||\mathcal{T}_H(u(x))|}\right)^2 h(u(x))\,\mathrm{d}x + \Theta,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ is non-negative, P is a uniformly elliptic operator in non-divergent form, $\mathcal{T}_H(\cdot)$ is certain transformation of the monotone C^1 function $H(\cdot)$, which is the primitive of the weight $h(\cdot)$, and Θ is the boundary term which depends on boundary values of u and ∇u. This inequality is motivated and closely connected to elliptic PDE and other applications.

References

[1] Kałamajska, A., Peša, D. & Roskovec, T., Non-linear GagliardoNirenberg inequality involving a second-order elliptic operator in non-divergent form, *Nonlinear Differ. Equ. Appl.*, **32**, 114(2025), https://doi.org/10.1007/s00030-025-01124-9